Contrôle d'erreur numérique a posteriori et critères d'arrêt pour des solveurs linéaires et non linéaires

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Two-phase flow in porous media

Two-phase flow in porous media

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\begin{aligned}
\partial_t(\phi s_\alpha) + \nabla\cdot\mathbf{u}_\alpha &= q_\alpha, && \alpha \in \{\mathrm{n}, \mathrm{w}\},\\
-\lambda_\alpha(s_\mathrm{w})\underline{\mathbf{K}}(\nabla p_\alpha + \rho_\alpha g \nabla z) &= \mathbf{u}_\alpha, && \alpha \in \{\mathrm{n}, \mathrm{w}\},\\
s_\mathrm{n} + s_\mathrm{w} &= 1,\\
p_\mathrm{n} - p_\mathrm{w} &= p_c(s_\mathrm{w})
\end{aligned}
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Mathematical issues

- coupled system
- unsteady, nonlinear
- elliptic–parabolic degenerate type
- dominant advection

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Theorem (A posteriori error estimate distinguishing the error components)

Let

- *n be the time step,*
- *k be the linearization step,*
- *i be the algebraic solver step,*

with the approximations $(s_{\mathrm{w},h\tau}^{n,k,i}, p_{\mathrm{w},h\tau}^{n,k,i})$. Then

$$|||(s_{\mathrm{w}} - s_{\mathrm{w},h\tau}^{n,k,i}, p_{\mathrm{w}} - p_{\mathrm{w},h\tau}^{n,k,i})|||_{I_n} \leq \eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i}.$$

Error components

- $\eta_{\mathrm{sp}}^{n,k,i}$: spatial discretization
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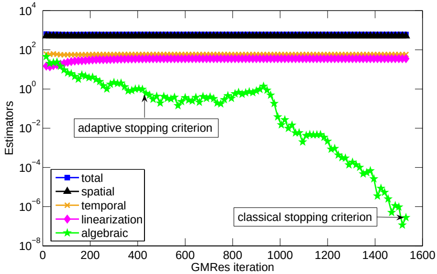
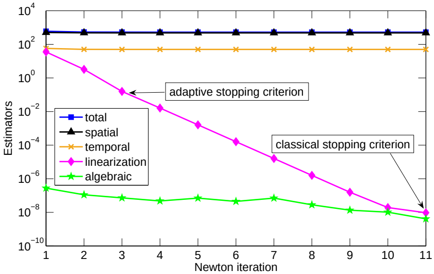
Estimators and stopping criteria



Estimators in function of GMRes iterations



Estimators in function of Newton iterations

Comments

- finite volumes, fully implicit time discretization
- we can stop much earlier and **economize many linear / nonlinear solver iterations**

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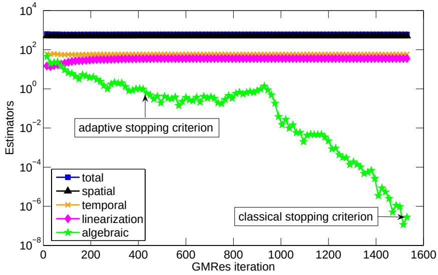

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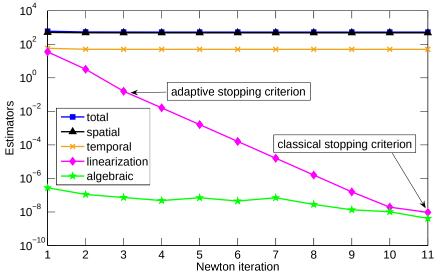

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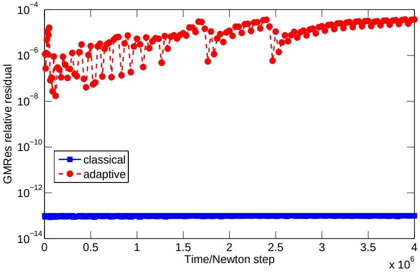
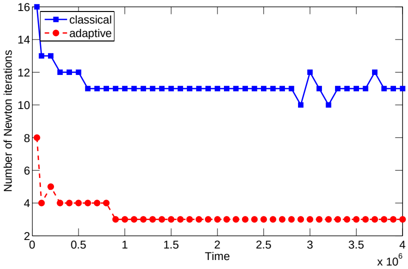
GMRes relative residual/Newton iterations

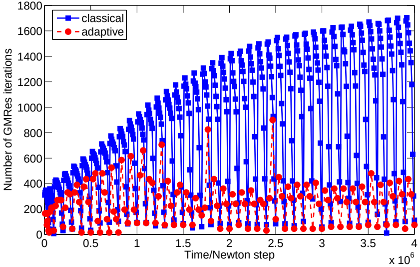


GMRes relative residual

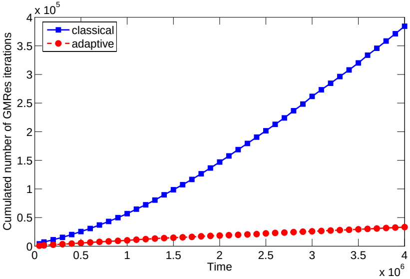


Newton iterations

GMRes iterations



Per time and Newton step



Cumulated

Nonlinear Laplacian with singular solution

Model problem

- p-Laplacian

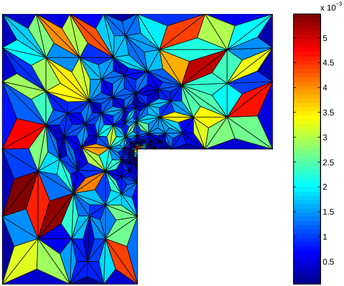
$$\begin{aligned} \nabla\cdot(|\nabla u|^{p-2}\nabla u) &= f \qquad \text{in } \Omega, \\ u &= u_0 \qquad \text{on } \partial\Omega \end{aligned}$$

- known weak solution (used to impose the Dirichlet BC)

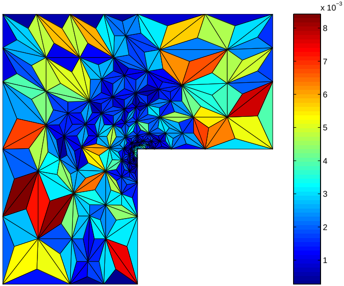
$$u(r,\theta) = r^{\frac{7}{8}} \sin(\theta \tfrac{7}{8})$$

- $p = 4$, L-shape domain, singularity in the origin
- nonconforming finite elements

Error distribution on an adaptively refined mesh



Estimated error distribution



Exact error distribution

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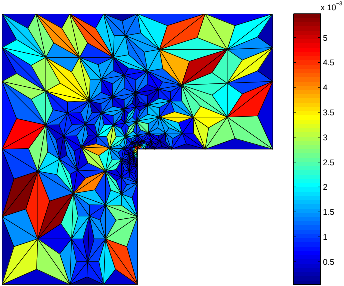

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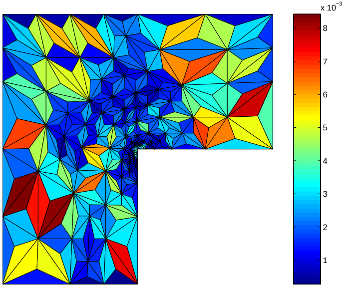

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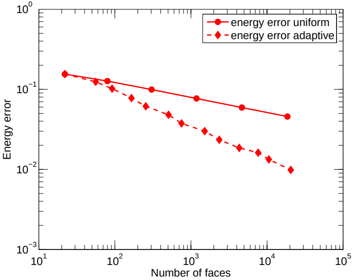
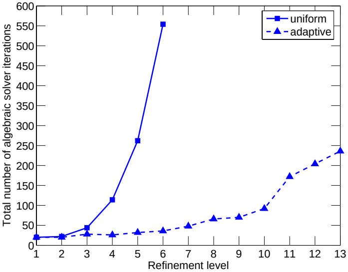
Energy error and overall performance



Energy error



Overall performance

Comments

- for the **same number of unknowns**, we obtain **much better precision** with adaptive mesh refinement (left)
- for the same number of unknowns, the calculation on **adaptively refined meshes** with **adaptive stopping criteria** is **cheaper** in total algebraic iterations (right)

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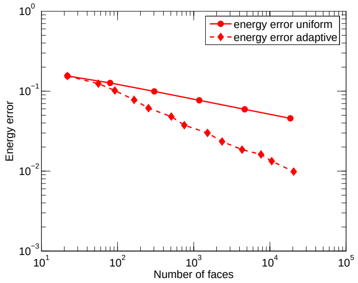

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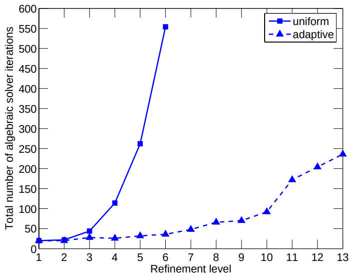

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Bibliography

Bibliography

- VOHRALÍK M., WHEELER M. F., A posteriori error estimates, stopping criteria, and adaptivity for two-phase flows, HAL Preprint 00633594.
- ERN A., VOHRALÍK M., Adaptive inexact Newton methods with a posteriori stopping criteria for nonlinear diffusion PDEs, *SIAM J. Sci. Comput.*, accepted for publication.